

GAUGE FIXING AND GLUON PROPAGATOR IN λ -GAUGES ^a

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We discuss critical slowing-down of several gauge-fixing algorithms for the so-called λ -gauges in the $SU(2)$ case at zero temperature. For these gauges we also evaluate the gluon propagator using different definitions of the lattice gluon field, corresponding to discretization errors of different orders.

1 Gauge Fixing

Efficiency of gauge fixing algorithms is an important issue in the study of gauge-dependent quantities such as the gluon propagator on the lattice. We study critical slowing-down (CSD) by determining the dynamic critical exponent z for various algorithms, using optimal gauge-fixing quantities.

We consider the $SU(2)$ case of the standard Wilson action and, in order to impose the λ -gauge condition¹, we look for a local minimum of the functional

$$\mathcal{E}_\lambda[g] \equiv - \sum_{\mathbf{x}} \text{Tr} \left[\lambda g(\mathbf{x}) U_d(\mathbf{x}) g^\dagger(\mathbf{x} + \mathbf{e}_d) + \sum_{\mu=1}^{d-1} g(\mathbf{x}) U_\mu(\mathbf{x}) g^\dagger(\mathbf{x} + \mathbf{e}_\mu) \right]. \quad (1)$$

This corresponds to the condition

$$\lambda \partial_d A_d^{(g)}(\mathbf{x}) + \sum_{\mu=1}^{d-1} \partial_\mu A_\mu^{(g)}(\mathbf{x}) = 0, \quad (2)$$

where A is the gluon field $(U - U^\dagger)/2i$. The case $\lambda = 1$ is the Landau gauge, whereas $\lambda = 0$ is the Coulomb gauge. At each step of a gauge-fixing algorithm, $g(\mathbf{x})$ is updated to a new value $g^{(new)}(\mathbf{x})$ in order to minimize $\mathcal{E}[g]$. We consider the following algorithms: ^{2,3,4} the *local* methods of Los Alamos (exact local minimization), Cornell (where the functional is minimized in the direction of the local downhill gradient of $\mathcal{E}[g]$), overrelaxation (where local minimization is combined with an “energy-preserving” update by means of a tuning parameter ω), and stochastic overrelaxation (where either type of update is randomly chosen at each step), as well as the *global* method of Fourier acceleration. This algorithm can be seen as a generalization of the Cornell method, whose update

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is given by

$$g^{(new)}(\mathbf{x}) \propto \left[\mathbb{1} - \alpha (\nabla \cdot A^{(g)})(\mathbf{x}) \right] g(\mathbf{x}) . \quad (3)$$

In fact, for the Fourier acceleration method we have^{3,4}

$$g^{(new)}(\mathbf{x}) \propto \left[\mathbb{1} - \alpha p_{max}^2 \Delta_\lambda^{-1} (\nabla \cdot A^{(g)})(\mathbf{x}) \right] g(\mathbf{x}) . \quad (4)$$

The inverse of the λ -Laplacian $\Delta_\lambda \equiv \lambda \partial_d^2 + \sum_{\mu=1}^{d-1} \partial_\mu^2$ is usually obtained from $\hat{F}^{-1} p_\lambda^{-2}(\mathbf{k}) \hat{F}$, where $p_\lambda^2(\mathbf{k})$ are the eigenvalues of Δ_λ , and \hat{F} indicates the Fourier transform. In the Landau case we have introduced³ a multigrid implementation of this method (where the inverse Laplacian is computed using a multigrid algorithm), showing improved behavior and applicability.

In order to monitor the convergence of these algorithms, we evaluate at each step the quantity $|\nabla \cdot A^{(g)}|^2$, as well as the quantity

$$Q \propto \sum_\nu \sum_c \sum_{x_\nu} [Q_\nu^c(x_\nu) - \overline{Q}_\nu^c]^2 / [\overline{Q}_\nu^c]^2 \quad (5)$$

where $\overline{Q}_\nu^c \equiv (1/N_\nu) \sum_{x_\nu} Q_\nu^c(x_\nu)$ and $Q_\nu^c(x_\nu) \equiv \sum_{\mu \neq \nu} \sum_{x_\mu} (A^{(g)})_\nu^c(\mathbf{x})$, which is seen to be particularly sensitive to the goodness of the gauge fixing². For these two quantities we expect to observe an exponential decay with the number of iterations t . In the limit of large t we can introduce a relaxation time τ such that the decay is given by $\exp(-t/\tau)$. As the lattice size N is increased, and at constant physics (see Section 1.1), τ will grow as $\tau \sim N^z$, where z is the dynamic critical exponent of the algorithm. [Note that all the algorithms we consider, except for Los Alamos, need tuning.]

In the Landau case we have simulated^{2,3,4} the above algorithms in the 2d case at finite values of the coupling β , and in the 4d case at $\beta = \infty$. We have found that, as expected, the Los Alamos algorithm has $z \sim 2$, the improved local algorithms (i.e. Cornell, overrelaxation and stochastic overrelaxation) have $z \sim 1$, and that $z \sim 0$ for Fourier acceleration. These results were also verified analytically⁴ at $\beta = \infty$. The multigrid implementation of Fourier acceleration is found to be considerably more stable and easier to tune than the standard method, and it can be used efficiently on parallel machines. Here we extend this study to the 4d case at finite β and to values of λ different from 1.

1.1 Simulations and Results

In order to analyze CSD for an algorithm, we have to evaluate τ for different combinations of lattice size N and coupling β , but at constant physics. This is done easily in $d = 2$ by keeping the product $N\sqrt{\sigma}$ constant, since the string

Table 1: Average number of iterations for the various gauge-fixing algorithms and for different λ -gauges. We consider $N = 8$ at $\beta = 2.6957$.

λ	LOS	COR	OVE	STO	FFTFA	MGFA
1	631(42)	155(11)	137(12)	159(8)	95(11)	94(10)
0.5	772(44)	176(21)	143(9)	178(10)	106(15)	93(6)

tension σ is a known function of β . No such function is available in $d = 4$. In this case we use the constant physics obtained⁵ by keeping a lattice definition of the running coupling α_s constant. This prescription is well described by the fit $\beta = 1.905 + 0.308 \ln N$ for $\beta \geq 8$. In particular, we consider $N = 8$ at $\beta = 2.6957$, $N = 12$ at $\beta = 2.8485$ and $N = 16$ at $\beta = 2.9586$. By fitting our data for the relaxation time τ to the function $\tau = cN^z$ we obtain the dynamic critical exponents z for the various algorithms (neglecting finite-size effects). Our previous results^{2,3,4} are confirmed also in the four-dimensional case at finite values of β : the Los Alamos method has $z \sim 2$, while the improved local algorithms have $z \sim 1$ and the Fourier acceleration method shows $z \sim 0$.

All these results were obtained using the stopping criterion $|\nabla \cdot A|^2 \leq 10^{-12}$. We also check how the performance of the various algorithms is modified if the condition $Q \leq 10^{-12}$ is used. For the case $N = 16$ and $\beta = 2.9586$ we have found, in agreement with our previous findings^{2,3,4}, that for the stochastic overrelaxation and Fourier acceleration methods the number of sweeps is essentially independent of the quantity chosen for the stopping criterion, while the Los Alamos, Cornell and overrelaxation methods require about 20% more sweeps if the quantity Q is considered.

Finally, in Table 1 we report the number of sweeps needed to satisfy the condition $|\nabla \cdot A|^2 \leq 10^{-12}$, obtained for lattice size $N = 8$ and coupling $\beta = 2.6957$ for the various algorithms for two different values of λ , namely 1 and 0.5. From these data we see that the performance of the algorithms seems to depend weakly on the value of λ .

2 Gluon Propagator

The study of the infrared behavior of the gluon propagator at zero temperature⁶ provides a powerful tool for gaining insight into the physics of confinement in non-Abelian gauge theories. In high-temperature QCD, the long-distance behavior of the gluon propagator is directly related to the electric and magnetic screening lengths^{7,8}. Here we study the gluon propagator at zero temperature, using different definitions of the lattice gluon field, and considering different λ -gauges. To this end, let us define the gluon propagator in

momentum space as

$$D(k) \equiv \frac{1}{9V} \sum_{\mu=1}^{d-1} D_{\mu}(k)$$

$$D_{\mu}(k) \equiv \sum_c \left[\sum_t \cos(2\pi kt) Q_{\mu}^c(t) \right]^2 + \left[\sum_t \sin(2\pi kt) Q_{\mu}^c(t) \right]^2 \quad (6)$$

where $Q_{\mu}^c(t) \equiv \sum_{x,y,z} A_{\mu}^c(x, y, z, t)$ and we set $\mathbf{k} = (0, 0, 0, k)$. For the lattice gluon field A_{μ} we consider several possible definitions, leading to discretization errors of different orders. For example, we define:

$$A_{\mu}^{(1)}(\mathbf{x}) \equiv \frac{U_{\mu}(\mathbf{x}) - U_{\mu}^{\dagger}(\mathbf{x})}{2i} \quad \text{and} \quad A_{\mu}^{(2)}(\mathbf{x}) \equiv \frac{[U_{\mu}(\mathbf{x})]^2 - [U_{\mu}^{\dagger}(\mathbf{x})]^2}{4i}. \quad (7)$$

If we set $U_{\mu}(\mathbf{x}) \equiv \exp[ia g_0 \vec{\sigma} \cdot \vec{A}(\mathbf{x})]$, we obtain that both $A_{\mu}^{(1)}(\mathbf{x})$ and $A_{\mu}^{(2)}(\mathbf{x})$ are equal to $ag_0 \vec{\sigma} \cdot \vec{A}(\mathbf{x})$ plus terms of order $a^3 g_0^3$. We can also consider

$$A_{\mu}^{(3)}(\mathbf{x}) \equiv 4/3 A_{\mu}^{(1)}(\mathbf{x}) - 1/3 A_{\mu}^{(2)}(\mathbf{x}) \quad (8)$$

and

$$A_{\mu}^{(4)}(\mathbf{x}) \equiv 64/45 A_{\mu}^{(1)}(\mathbf{x}) - 20/45 A_{\mu}^{(2)}(\mathbf{x}) + 1/45 \frac{[U_{\mu}(\mathbf{x})]^4 - [U_{\mu}^{\dagger}(\mathbf{x})]^4}{8i}. \quad (9)$$

It is easy to check that $A_{\mu}^{(3)}(\mathbf{x}) = ag_0 \vec{\sigma} \cdot \vec{A}(\mathbf{x})$ plus terms of order $a^5 g_0^5$, and that $A_{\mu}^{(4)}(\mathbf{x}) = ag_0 \vec{\sigma} \cdot \vec{A}(\mathbf{x})$ plus terms of order $a^7 g_0^7$. The definitions $A^{(1)}$ and $A^{(2)}$ were recently considered by Giusti et al⁹. They find that the corresponding gluon propagators are equal modulo a constant factor. We perform a similar study here, evaluating $D^{(i)}(k)$ using the different definitions of the gluon field $A_{\mu}^{(i)}$ given above, and we try to give an interpretation to this constant factor.

Notice that $D_1^{(i)}(k) + D_2^{(i)}(k)$ is related to the gluon propagator used by Karsch et al^{7,8} for the evaluation of the magnetic screening mass, and that this screening mass is invariant under rescaling of the propagators by a constant factor.

2.1 Simulations and Results

We have performed simulations at several values of β , for lattice volumes $V = 8^4, 12^4$, and for the gauge parameter $\lambda = 1$ (Landau gauge) and $\lambda = 0.5$. We obtain that, in all cases, the four propagators $D^{(i)}(k)$ are equal modulo a constant factor. Let us notice that this proportionality constant between different discretizations of the gluon propagator may be explained [at least for the simple cases $D^{(1)}(k)$ and $D^{(2)}(k)$] as a tadpole renormalization¹⁰. In fact,

let us consider the tadpole-improved link $\tilde{U}_\mu(\mathbf{x}) \equiv U_\mu(\mathbf{x})/u_0$, where u_0 is the mean link in Landau gauge. Then $D^{(1)}(k)$ gets multiplied by a factor u_0^{-2} , and $D^{(2)}(k)$ by u_0^{-4} . At $\beta = 2.2$ and $\lambda = 1$ we have $u_0^2 = 0.6790(1)$. Thus, using tadpole-improved operators, the discrepancy $D^{(1)}(k)/D^{(2)}(k)$ is reduced from 1.862(9) to 1.264(6). (A similar analysis can be done at $\lambda = 0.5$.)

The gluon propagator is a gauge-dependent quantity. At finite temperature, however, the pole masses obtained from the exponential decay of the gluon correlation functions at large spatial separations were proven to be gauge-independent, and therefore gluon propagators in different gauges should be related by a constant factor, as found in ⁸. Here we consider only the zero-temperature case, and we expect to observe gauge dependence. We have checked that this is indeed the case, i.e. we observe gauge dependence for each given discretization $D^{(i)}$. Nevertheless, we see that, within a gauge, the behavior of the gluon propagator is independent of the discretization modulo a renormalization factor. We plan to extend this study to the case of QCD at high temperature. A similar behavior in that case would mean that the screening masses are independent of the definition used for the lattice gluon field.

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